

HYPERCUBE EMULATION OF INTERCONNECTION NETWORKS TOPOLOGIES

ADEL ALAHMADI, HUSAIN ALHAZMI, SHAKIR ALI, MICHEL DEZA,
MATHIEU DUTOIR SIKIRIĆ, AND PATRICK SOLÉ

ABSTRACT. We address various topologies (de Bruijn, chordal ring, generalized Petersen, meshes) in various ways (isometric embedding, embedding up to scale, embedding up to a distance) in a hypercube or a half-hypercube. Example of obtained embeddings: infinite series of hypercube embeddable Bubble Sort and Double Chordal Rings topologies, as well as of regular maps.

1. INTRODUCTION

The hypercube topology is a very popular topology for Parallel Processing computers from the Connection Machine [12] onward. One way to emulate an alternative topology on such a computer is to address the vertices of the guest topology by the vertex of the host hypercube or some subgraph thereof. This addressing can be used for routing purposes, for instance. Another important application is the addressing of knowledge databases [16]. This latter application is important for natural language processing.

In the present work we model graph theoretically the addressing process in various ways from isometric embedding (the guest graph is a so-called *partial cube*) to embedding up to scale (geodetic distance on the host is a constant times that of the guest) or up to a given distance (called henceforth *truncated embedding*). This work is an application, a continuation and a generalization of the book [6], which considers only embeddings. We shall consider many popular topologies in turn and will question their embeddability. The material is organised as follows. To begin with, we consider insertion/deletion-based distances in Section 4 and other graphs defined on alphabets (Odd graph, Generalized Petersen, De Bruijn) in Section 5 and move on to cycle-based topologies in Section 8. Hypercube based topologies (Cube-connected Cycles, Butterfly graphs) in Section 6 and Cayley graphs on the group of permutations in Section 7 are also considered. We conclude in Section 9 by regular maps: skeletons of Klein graph, Dyck graph and so on.

2. PRELIMINARIES

Denote by H_m the skeleton of the m -dimensional cube. It is the graph on all binary sequences of length m with two of them, say, $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, being adjacent if their *Hamming distance*

$$d_H(x, y) = \sum_{i=1}^m |x_i - y_i|$$

is 1. Clearly, Hamming distance is an l_1 -metric and the square of l_2 -metric (Euclidean distance) on these sequences.

Denote by $\frac{1}{2}H_m$ the *m-half-cube graph*. It is defined on all binary sequences of length m having even number of ones, with two of them, say, x and y , being adjacent if $d_H(x, y) = 2$. Clearly, H_m is an isometric subgraph of $\frac{1}{2}H_{2m}$.

Given a finite connected graph $G = (V, E)$ of diameter d , let $D(G)$ denote the (shortest path) distance matrix $((d_{ij}))$ of its vertices. Call G *embeddable* (or, as it is done in [3], *code graph*) and denote it by $G \rightarrow \frac{1}{2}H_m$, if G is an isometric subgraph of a some m -half-cube, i.e., $((d_{ij}))$ embeds *scale-2-isometrically* into the distance matrix of m -cube. If, moreover, G is an isometric subgraph of a m -cube, denote it by $G \rightarrow H_m$ and call G a *partial cube*. Clearly,

$$\text{if } G \rightarrow \frac{1}{2}H_m \text{ and } G' \rightarrow \frac{1}{2}H_{m'}, \text{ then } G \times G' \rightarrow \frac{1}{2}H_{m+m'}.$$

Another isometric subgraph of $\frac{1}{2}H_m$ is the *Johnson graph* $J(m, k)$; its vertices are the k -element subsets of an m -element set, and two vertices are adjacent when they meet in a $(k - 1)$ -element set. Let us denote by $G \rightarrow J(m, k)$ such eventual special case of embedding into $\frac{1}{2}H_m$.

Theorem 1. ([17])

For a connected graph G , it holds:

(i) G is l_1 -embeddable (i.e., it embeds isometrically into some l_1 -space) if and only if $D(G)$, for some integers $m, \lambda \geq 1$, embeds *scale- λ -isometrically* into the distance matrix of m -cube;

(ii) if G is l_1 -embeddable, then it is an hypermetric graph, i.e., its d_G satisfies all hypermetric inequalities

$$\sum_{1 \leq i < j \leq n} b_i b_j d_G(v_i, v_j) \leq 0,$$

where $b = \{b_1, b_2, \dots, b_n\} \in \mathbb{Z}^n$, $\sum_{i=1}^n b_i = 1$ and v_i , $1 \leq i \leq n$, are vertices of G .

The inequality with $\sum_{i=1}^n |b_i| = 2k + 1$ is called a $(2k + 1)$ -gonal inequality. Clearly, the case $k = 1$ corresponds to the usual triangle inequality. The *5-gonal inequality* correspond to $b_a = b_b = b_c = 1$, $b_x = b_y = -1$, i.e., it is

$$d(x, y) + (d(a, b) + d(a, c) + d(b, c)) \leq \sum_{i=a,b,c} (d(x, i) + d(y, i))$$

for any vertices a, b, c, x, y . [2] showed that a connected graph is a partial cube if and only if it is bipartite and its path-metric satisfy all 5-gonal inequalities. See examples of not 5-gonal graphs on Fig. 1.

The hypermetricity is not sufficient, if the number of vertices is greater than 6, for embeddability and, larger, for l_1 -embeddability; see Fig. 2.

Theorem 2. (Theorem 17.1.1 in [7]) For a connected graph G , it holds:

(i) G is hypermetric if and only if it is an isometric subgraph of a Cartesian product of half-cube graphs $\frac{1}{2}H_m$, cocktail-party graphs $K_{2, \dots, 2}$ and copies of the Gosset graph G_{56} ;

(ii) G is an l_1 -graph if and only if it is an isometric subgraph of a Cartesian product of half-cube graphs $\frac{1}{2}H_m$ and cocktail-party graphs $K_{2, \dots, 2}$.

Given an integer $2 \leq s \leq d$, call G *s-tr.embeddable* (short for *up to s truncated-embeddable*) if there exists a distance matrix $D' = ((d'_{ij}))$ of order $|V|$ with $d'_{ij} = d_{ij}$, whenever $d_{ij} \leq s$, which is isometrically embeddable in the distance matrix of some m -half-cube. So, D' is a graphic distance matrix only if $D' = D$.

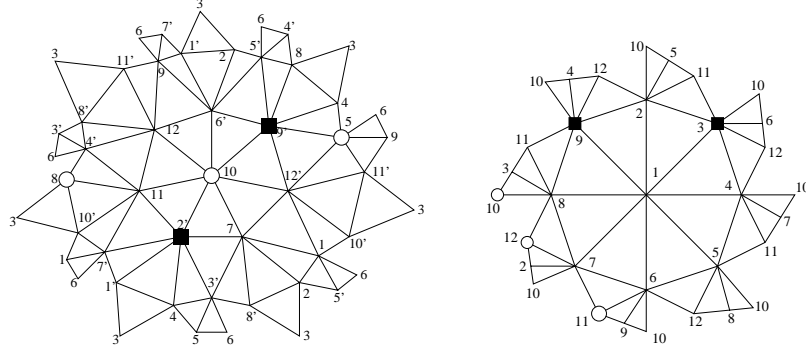


FIGURE 1. Examples of not 5-gonal regular maps: dual Klein map $\{7, 3\}$ and dual Dyck map $\{8, 3\} \simeq K_{4,4,4}$ on genus 3 surface

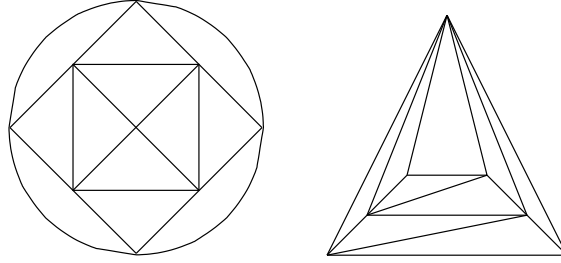


FIGURE 2. Examples of hypermetric, but not embeddable graphs

Clearly, $0 \leq d'_{ij} \leq d_{ij}$, whenever $d_{ij} > s$, and s -tr.embeddability implies $(s - 1)$ -tr.embeddability. Such embedding produces an addressing of vertices by binary sequences, preserving, up to scale 2 and up to value s the graph's distances. This addressing can be used for routing purposes and for the emulation of an architecture of topology G on a host machine that has a hypercube topology, as, say, the Connection Machines from CM-1 to CM-4.

The classical structures, used as topologies for interconnection networks, are, say, trees, hypercubes and rings. First two, as well as even rings, are, trivially, partial cubes. An odd ring C_n embeds into $\frac{1}{2}H_n$. Here we consider eventual embedding for other topologies, especially, for those mentioned in a good surveys [12], [9]. This work, while being a follow-up of the book [6], since we again investigate embeddability, is focused now on network applications. Also, in the case of absence of embedding, we look now for "embedding" in weaker sense, i.e., maximal $s < d$, for which s -tr.embedding eventually exist. We not consider the setting of l_1 -embedding, because it looks as not suitable for applications.

3. ALGORITHM

Our work is based on heavy computations, using programs based on algorithm in [8]. This algorithm (with time complexity $O(n^2 + nm)$ and space complexity

$O(n^2)$) constructs an embedding into H_m up to scale 2, if one exists. The method has been extended to scale 2 embeddings up to a given distance s .

For any edge $e = \{v, v'\}$ of G and any such embedding ϕ , the difference $\phi(v) - \phi(v')$ corresponds to a set S_e of length 2. Lemma 4.1 in [8] allows in some cases to compute the size $i(e, e')$ of the intersection $S_e \cap S_{e'}$ for two edges e and e' . One of the conditions of applicability of the lemma is that vertices in e and e' are at distance at most s .

In the case of $s = \text{diam}(G)$, we can take a spanning tree T of G and compute $i(e, e')$ for all pairs of edges e, e' . If the function i is negative, then the graph is not embeddable. Otherwise, we can identify edges, such that $i(e, e') = 2$, and check that this defines an equivalence relation. Afterwards we define a graph H on classes \bar{e} and \bar{e}' with two classes adjacent if $i(e, e') = 1$. We then check if the graph admits an inverse line graph by implementing the algorithm of [15].

In some cases the embedding is not unique; see, for example, Tetrahedron in Figure 7. All such cases of non-unique reversed line graph are classified in [15].

In the general case of s -tr.embedding, we may not have computed all the values $i(e, e')$. However, in the case of s -tr.embedding, one has $i(e, e') \in \{0, 1, 2\}$ and also following consistency relations:

- (1) If $i(e_1, e_2) = 2$, then for any other edge e' , it holds $i(e_1, e') = i(e_2, e')$.
- (2) If e_1, e_2, e_3 are three edges with $i(e_i, e_j) = 1$ for $i \neq j$, then the edges e_1, e_2 and e_3 can be of the form:

$$e_1 = AB, e_2 = AC, e_3 = BC.$$

In that case, for any other edge e , we will have, up to permutation, following patterns of intersections: $\{i(e_1, e), i(e_2, e), i(e_3, e)\} : \{1, 1, 0\}, \{1, 1, 2\}$ and $\{0, 0, 0\}$. Alternatively, the edges e_1, e_2 and e_3 can be of the form:

$$e_1 = AB, e_2 = AC, e_3 = AD.$$

In that case, for any other edge e , there is, up to permutation, following patterns of intersections: $\{i(e_1, e), i(e_2, e), i(e_3, e)\} : \{1, 0, 0\}, \{1, 1, 0\}, \{1, 1, 1\}, \{1, 1, 2\}$ and $\{0, 0, 0\}$. Also, the number of patterns $\{1, 1, 0\}$ is at most 3.

At start, when Lemma 4.1 of [8] can be applied, we set up $i(e, e')$. If it cannot be applied, we only know that $i(e, e') \in \{0, 1, 2\}$. With the above relations, one can sometimes deduce $i(e, e')$ from what is known and this can be iterated. Therefore, in some cases the values of $i(e, e')$ is completely determined from the values obtained from Lemma 4.1 in [8]. But in other cases, the above logical relations are not sufficient to deduce all possible values $i(e, e')$. Thus, we apply a classical backtracking strategy of choosing the value $i(e, e')$, applying above deduction rules and iterating until we find all possible embeddings.

4. INDEL-BASED GRAPHS

Denote by D_n the set of binary sequences of length n and, for any $0 \leq i \leq n$, denote by $D_{i, \dots, n}$ the set $\cup_{j=i}^n D(j)$.

The *indel graph* $\text{Ind}_{i, \dots, n}$ is defined on $D_{i, \dots, n}$ by considering two sequences adjacent if one can be obtained from the other by *indels*, i.e., insertions or deletions of characters only. This graph is bipartite and has diameter $2n$.

| vertex | vertex address |
|--------------|----------------------|
| 0,0 | (0,0,0,0,0,0) |
| 1,0 | (0,1,1,0,0,0) |
| 1,1 | (0,1,1,1,1,0) |
| 0,1 | (0,0,1,0,1,0) |
| 0,0,0 | (1,0,0,0,0,0) |
| 1,0,0 | (0,1,0,0,0,0) |
| 0,1,0 | (0,0,1,0,0,0) |
| 1,1,0 | (0,1,1,1,0,0) |
| 1,1,1 | (0,0,1,1,1,1) |
| 1,0,1 | (0,1,1,0,1,0) |
| 0,1,1 | (0,0,1,1,1,0) |
| 0,0,1 | (0,0,0,0,1,0) |

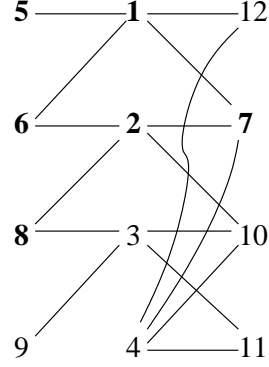


TABLE 1. Embedding $Ind_{2,3} \rightarrow H_6$. The boldfaced minor of rows 1, 2, 5 – 8 and columns 1 – 4 gives embedding $Ind_{1,2} \rightarrow H_4$. The minor of rows 1, 2, 5 and columns 1, 2 gives embedding $Ind_{0,1} \rightarrow H_2$

Proposition 1. *With exception of $Ind_{0,1} = P_2 \rightarrow H_2$ and $Ind_{n-1,n} = H_{2n}$ for $n = 2, 3$, any $Ind_{i,\dots,n}$ is not 2-tr.embeddable.*

Proof. In fact, see three embeddings on Table 1. Clearly, $Ind_{n-1,n}$ is an isometric subgraph of $Ind_{n,n+1}$ and, for $0 \leq i \leq n-2$, of $Ind_{i,\dots,n}$. By computation, $Ind_{0,1,2}$ and $Ind_{3,4}$ are not 2-tr.embeddable, proving the Proposition. \square

The *Levenshtein graph* $Lev_{i,\dots,n}$ is defined on $D_{i,\dots,n}$ by considering two sequences adjacent if one can be obtained from the other by indels and changes of characters as say, x on $1-x$, only. This graph has diameter n .

Proposition 2. *With exception of $Lev_{0,1} = K_3 \rightarrow \frac{1}{2}H_3$, any $Lev_{i,\dots,n}$ is not 2-tr.embeddable.*

Proof. In fact, $Lev_{n-1,n}$ is an isometric subgraph of $Lev_{n,n+1}$ and, for $0 \leq i \leq n-2$, of $Lev_{i,\dots,n}$. By computation, $Lev_{1,2}$ is not 2-tr.embeddable, proving the Proposition. \square

Note, that [1] gave lower bound $\frac{3}{2}$ for distortion of a *Lipchitz l_1 -embedding* of Levenshtein metric on sequences; see also Problem 2. 15 in [13]. Given metric spaces (X, d_X) and (Y, d_Y) , the *distortion* of a mapping $f : X \rightarrow Y$ is $\|f\|_{Lip}\|f^{-1}\|_{Lip}$, where the *Lipschitz norm* is defined by

$$f_{Lip} = \sup_{x,y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

The *Ulam metric* (or *permutation editing metric*) U is an editing metric on Sym_n , obtained by *character moves*, i.e., transpositions of characters. It is the half of the indel metric on Sym_n and It is right-invariant. Also, $n - U(x, y) = LCS(x, y)$, where $LCS(x, y)$ is the length of the longest common subsequence (not necessarily a substring) of x and y . The *Ulam graph* $Ul(n)$ has diameter $n-1$.

Proposition 3. *With exception of $Ul(2) = K_2 = H_1$ and $Ul(3) = K_{2,2,2} \rightarrow \frac{1}{2}H_4$, any $Ul(n)$ is not 2-tr.embeddable.*

| (n,k) | graph's name if any | diameter | embedding into |
|------------|---------------------|----------|-----------------------|
| (n=2m,1) | $Prism_{2m}$ | m+1 | H_{m+1} |
| (n=2m+1,1) | $Prism_{2m+1}$ | m+1 | $\frac{1}{2}H_{2m+3}$ |
| (10,3) | Desargues graph | 5 | H_5 |
| (10,2) | Dodecahedron | 5 | $\frac{1}{2}H_{10}$ |
| (9,2) | | 4 | $\frac{1}{2}H_9$ |
| (6,2) | Dürer octahedron | 4 | $\frac{1}{2}H_8$ |
| (5,2) | Petersen graph | 2 | $\frac{1}{2}H_6$ |

 TABLE 2. The cases of embedding for Generalised Petersen graph $GP(n,k)$

Proof. In fact, see the embedding of $Ul(3)$, i.e., Octahedron, in Fig. 7 Clearly, $Ul(n)$ is an isometric subgraph of $Ul(n+1)$. By computation $Ul(4)$ is not 2-tr.embeddable, proving the Proposition. \square

5. NETWORK GRAPHS ON ALPHABETS

Here we consider some graphs, where the vertices are labeled by words of length n over an alphabet and their relatives.

The *Odd graph* O_n has one vertex for each of the $(n-1)$ -element subsets of a $(2n-1)$ -element set; two vertices are connected by an edge if and only if the corresponding subsets are disjoint. Any O_n is a distance-transitive graph of diameter $n-1$.

The Petersen graph is O_3 . Any O_n with $n \geq 4$ is even not 3-tr.embeddable, since O_4 is not embeddable and any O_{n-1} is an isometric subgraph of O_n . But for the bipartite double of O_n , called *Double Odd graph* (or *revolving doors*) DO_{2n-1} , it holds $DO_{2n-1} \rightarrow H_{2n-1}$; together with hypercubes and even cycles, those graphs are only distance-regular ones ([11]), which are partial cubes. Note that DO_5 is the Desargues graph $GP(10, 3)$.

The *Generalized Petersen graph* $GP(n, k)$ is (Coxeter, 1950) a graph consisting of an inner star polygon $\{n, k\}$ and an outer regular polygon $\{n\}$ with corresponding vertices in the inner and outer polygons connected with edges. For example, $GP(5, 2)$, $GP(8, 3)$ and $GP(12, 5)$ are well-known *Petersen graph*, *Möbius–Kantor graph* and *Nauru graph*, respectively.

All case of embeddable $GP(n, k)$ are given in Table 2. Möbius–Kantor graph and Nauru graph are not embeddable and, moreover, not 3-tr.embeddable.

A *Moore graph* is a graph of diameter d with girth $2d+1$. The Moore graphs are: $K_n(n > 2)$, C_{2n+1} , the Petersen graph, the *Hoffman–Singleton graph* HS (diameter 2, girth 5, degree 7, order 50) and a hypothetical graph of diameter 2, girth 5, degree 57 and order 3,250. We found that HS is not embeddable.

The undirected *De Bruijn graph* $Br(m, n)$ is a graph on m^n n -tuples $(a_1 \dots a_n)$ over a m -character alphabet (denoted by juxtaposition). The edges are defined to be pairs of the form $((a_1 \dots a_n), (a_2 \dots a_n a_{n+1}))$, where a_{n+1} is any character in the alphabet. The undirected *Kautz graph* $Ka(m, n)$ is defined similarly, but only tuples $(a_1 \dots a_n)$ with $a_i \neq a_{i+1}$ for each i are taken. The diameters of $Br(m, n)$ and, for $m \geq 3$, $Ka(m, n)$ are n . Clearly,

$$Br(2, 2) = K_4 - e \rightarrow \frac{1}{2}H_4 \quad \text{and} \quad Ka(2, n) = K_2 = H_1.$$

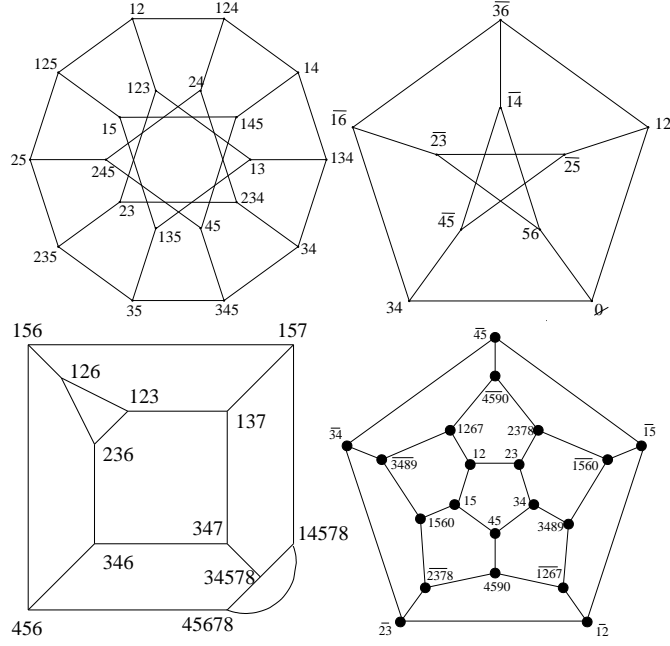


FIGURE 3. Embedding of the Desargues Graph $GP(10,3)$ into H_5 and of the Petersen graph $GP(5,2)$, Dürer's octahedron $GP(6,2)$ and Dodecahedron $GP(10,2)$ into $\frac{1}{2}H_6$, $\frac{1}{2}H_8$, $\frac{1}{2}H_{10}$, respectively

Conjecture 1. (i) all $Br(m,n)$ with $(m,n) \neq (2,2)$ are not 2-tr.embeddable; we checked it for $(m,n) = (3,2), (4,2), (5,2), (2,3), (3,3), (4,3), (5,3), (2,4), (3,4), (4,4), (2,5), (3,5), (3,6)$;
(ii) all $Ka(m,n)$ with $m \geq 3$ are not 2-tr.embeddable; we checked it for $(m,n) = (3,2), (4,2), (5,2), (6,2), (3,3), (4,3), (5,3), (3,4), (4,4), (5,4)$.

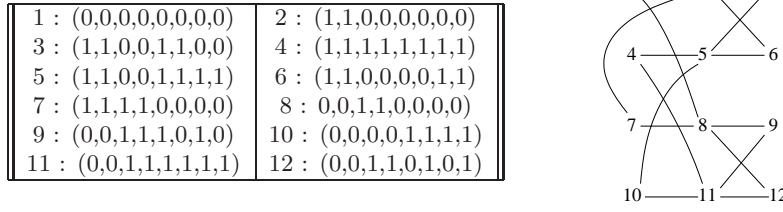
6. HYPERCUBE STRUCTURES

The *Cube-connected Cycles* CCC_n is [14] cubic graph, formed by replacing each vertex of an n -cube graph by a n -cycle. So, for example, CCC_3 is Truncated Cube. The diameter of CCC_n is 6 for $n = 3$ and $\lfloor \frac{5n-4}{2} \rfloor$ for $n \geq 4$; note that value $\lfloor \frac{5n-2}{2} \rfloor$, given in Table 3.4 of [12], is correct only for $n = 3$.

Conjecture 2. (checked $n = 3, 4, 5$)
 CCC_n is not $(n+1)$ -tr.embeddable.

CCC_3 is the only planar CCC_n ; its dual embeds into $\frac{1}{2}H_{12}$. In fact (cf. [6]), for any semiregular polyhedron P (i.e., one of 13 Archimedean polyhedra, prisms and antiprisms), exactly one of skeletons of P and its spherical dual P^* is embeddable.

A *Generalized Boolean n -cube* $GQ(r,n)$, defined on p. 28 of [12], is the direct product $C_r \times H_n$. So, $C_r \times H_n \rightarrow \frac{1}{2}H_{r+2n}$ and, for even r , $C_r \times H_n \rightarrow H_{\frac{r}{2}+n}$.


 FIGURE 4. Butterfly graph $But(2)$ and a 3-embedding $But(2) \rightarrow \frac{1}{2}H_8$

A *Mesh* and a *Generalised Hypercube* are direct products of paths and of complete graphs, respectively. Clearly, it holds

$$(P_{m_1} \times \cdots \times P_{m_k}) \rightarrow H_{(m_1 + \cdots + m_k) - k} \quad \text{and} \quad (C_{m_1} \times \cdots \times C_{m_k}) \rightarrow \frac{1}{2}H_{(m_1 + \cdots + m_k)}.$$

The undirected *Butterfly Graph* $But(n)$ is (cf., for example, p. 12 in [18]) a graph on $2^n(n+1)$ pairs (x, i) , where x is a binary sequence of length n and $i \in \{0, 1, \dots, n\}$, with vertices (w, i) and $(w', i+1)$ being adjacent if w' is identical to w in all bits with the possible exception of the $(i+1)$ -th bit counted from the left. (Note that the definition of Butterfly Graph in [9] is slightly different: it has 2^{2n} vertices there.) The diameter of $But(n)$ is $2n$. It holds $But(1) = C_4 = H_2$, while $But(2)$ and $But(3)$ are not 4-tr. embeddable. Still $But(2)$ admits nine 3-tr. embeddings into $\frac{1}{2}H_8$; see one of them on Table 4. Each column of this 12×8 binary matrix has exactly 6 ones.

The *Fibonacci Cube* $Fi(n)$ is the subgraph of H_n induced by the binary *Fibonacci sequences*, i.e., those containing no two consecutive ones. The *Lucas cube* $Lu(n)$ is the subgraph of H_n induced by Fibonacci sequences x_1, \dots, x_n such that not both x_1 and x_n are equal to 1. Both, $Fi(n)$ and $Lu(n)$, are partial cubes; cf. [10].

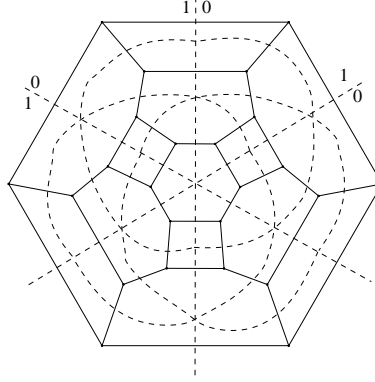
7. CAYLEY GRAPHS ON $Sym(n)$

Given a finite group G and a generating set S with $S = S^{-1}$ and $id \notin S$, the *Cayley graph* $CG(G, S)$ having G as the vertex-set and the edge-set consists of pairs of the form (g, gs) , with $g \in G, s \in S$. Most of the vertex-transitive structures - m -cubes, Generalised m -cubes, Cube-Connected-Cycles (but not Petersen graph) - are Cayley graphs. Below G will always be the symmetric group $Sym(n)$.

The *Star graph* $SG(n)$ is the Cayley graph with $S = \{(1, 2), (1, 3), \dots, (1, n)\}$; its diameter is $\left\lfloor \frac{3(n-1)}{2} \right\rfloor$. It holds $SG(3) = BSG(3)$. We conjecture that $SG(n)$ with $n \geq 4$ is not 3-tr. embeddable and checked it for $n = 4, 5$ and 6 .

The *Bubble Sort graph* $BSG(n)$ is the Cayley graph with $S = \{(1, 2), (2, 3), \dots, (n-1, n)\}$; its diameter is $\binom{n}{2}$. Its geodesic metric is called *Kendall τ distance* $I(x, y)$. (or *inversion metric*, *permutation swap metric*). It is an editing metric on $Sym(n)$: the number of adjacent transpositions needed to obtain x from y . Also, $I(x, y)$ is the number of *relative inversions* of x and y , i.e., pairs $(i, j), 1 \leq i < j \leq n$, with $(x_i - x_j)(y_i - y_j) < 0$.

Proposition 4. *For any Bubble Sort graph, it holds $BSG(n) \rightarrow H_{\binom{n}{2}}$.*

FIGURE 5. Embedding of Bubble Sort graph: $BS(4) \rightarrow H_6$

Proposition 1 in [5] (cf. also Table 3 there) shows that, given a finite Coxeter group W and its canonical generating set S , the Cayley graph $Cay(W, S)$ is isometrically embeddable into $H_{|T|}$, where T is the set of elements, that are conjugate to an element of S . Above Proposition is just the case $W = A_{n-1}$, since $Sym(n)$ is isomorphic to the finite Coxeter group A_{n-1} .

The *Pancake graph* $Pc(n)$ is the Cayley graph with S consisting of $n - 1$ permutations of the form $(i, i - 1, \dots, 1, i + 1, \dots, n)$; cf. [9]. It holds

$$Pc(3) = C_6 \rightarrow H_3.$$

We conjecture that for $n \geq 4$, $Pc(n)$ is not 3-tr.embeddable and checked it for the cases $n = 4, 5, 6$ with diameters 4, 5 and 6, respectively. To find the diameter of $Pc(n)$ in general, is an open problem, called the *prefix reversal problem*.

The *Swap-or-Shift graph* SOS_n^n is the Cayley graph with S consisting the shift $(1, \dots, n)$ and transposition $(1, 2)$. The graph SOS_n^{n-1} is the Cayley graph with S consisting the shift $(2, \dots, n)$ and transposition $(1, 2)$. It holds

$$SOS_3^3 = Prism_3 \rightarrow \frac{1}{2}H_5 \text{ and } SOS_4^4 \rightarrow \frac{1}{2}H_{12},$$

but SOS_5^5 , having diameter 10, is not 6-tr.embeddable. It holds

$$SOS_3^2 = C_6 \rightarrow H_3,$$

but SOS_4^3 , having diameter 6, is not 5-tr.embeddable, but it admits four 4-tr.embeddings into $\frac{1}{2}H_{14}$. Also, SOS_5^4 , having diameter 9, is not 5-tr.embeddable.

So, we expect that SOS_n^n is not 6-tr.embeddable for $n \geq 5$ and SOS_n^{n-1} is not 5-tr.embeddable for $n \geq 4$.

8. GRAPHS ON CYCLES

The *Möbius ladder* M_{2m} is a cubic circulant graph with $2m$ vertices, formed from an m -cycle by adding edges connecting opposite pairs of vertices in the cycle. We conjecture that M_{2m} is not 2-tr.embeddable and checked it for the *Thomsen* (or *utility*) graph $M_6 = K_{3,3}$, Wagner graph M_8 (both of diameter 2) and for M_{10} of diameter 3.

| | |
|----------------------|----------------------|
| ABCD : (0,0,0,0,0) | DBCA : (0,0,1,1,1,0) |
| DACB : (1,1,0,1,0,1) | DBAC : (1,0,0,0,0,1) |
| ADCB : (1,1,0,1,1,1) | BDCA : (0,1,1,1,1,0) |
| BACD : (1,0,0,0,0,0) | BDAC : (1,1,0,0,0,1) |
| ADBC : (0,0,1,0,1,0) | CDBA : (1,0,0,0,1,1) |
| CABD : (0,1,1,1,0,1) | CDAB : (0,1,1,0,0,0) |
| ABDC : (0,1,1,1,1,1) | CBDA : (0,1,0,0,0,1) |
| CADB : (1,0,1,0,1,0) | CBAD : (1,0,1,1,1,1) |
| ACDB : (1,0,0,0,1,0) | BCDA : (0,1,0,0,0,0) |
| BADC : (1,1,1,1,1,1) | BCAD : (1,0,1,1,1,0) |
| ACBD : (0,1,0,1,0,1) | DCBA : (1,0,0,1,1,1) |
| DABC : (0,0,1,0,0,0) | DCAB : (0,1,1,1,0,0) |

| | |
|--------------------------------------|--------------------------------------|
| ABCD : (0,0,0,0,0,0,0,0,0,0,0,0,0,0) | DBCA : (1,0,1,0,0,0,0,0,1,0,1,0,1,1) |
| DACB : (0,0,0,0,1,1,0,0,0,1,1,0,1,1) | DBAC : (1,0,1,0,0,0,1,1,1,0,1,0,1,1) |
| ADCB : (1,1,0,0,0,0,1,1,1,0,1,0,0,0) | BDCA : (0,0,0,0,1,1,1,1,0,0,1,1,0,0) |
| BACD : (1,0,0,1,1,1,1,1,0,1,0,1,1) | BDAC : (0,0,0,0,1,1,0,0,0,0,1,1,0,0) |
| ADBC : (1,1,0,0,0,0,1,1,1,0,1,0,1,1) | CDBA : (1,0,1,0,0,0,0,0,0,1,1,0,1,1) |
| CABD : (1,1,0,0,0,0,0,0,0,0,0,0,0,0) | CDAB : (1,0,1,0,1,1,0,0,0,1,1,0,1,1) |
| ABDC : (0,0,0,0,1,1,0,0,0,0,0,0,0,0) | CBDA : (1,0,0,1,1,1,1,1,0,0,1,1,0,0) |
| CADB : (1,1,0,0,0,0,1,1,0,0,0,0,0,0) | CBAD : (1,0,0,1,1,1,1,1,0,0,1,1,1,1) |
| ACDB : (1,0,0,1,1,1,0,0,0,1,1,0,1,1) | BCDA : (1,0,1,0,0,0,0,0,0,0,0,0,1,1) |
| BADC : (1,0,0,1,0,0,1,1,1,0,1,0,1,1) | BCAD : (1,0,1,0,0,0,0,0,0,0,0,0,0,0) |
| ACBD : (1,0,0,1,1,1,1,1,0,1,1,0,1,1) | DCBA : (1,1,0,0,1,1,1,1,0,0,1,1,0,0) |
| DABC : (0,0,0,0,1,1,0,0,0,1,1,0,0,0) | DCAB : (1,1,0,0,0,0,1,1,0,0,1,1,0,0) |

 TABLE 3. Embedding $SOS_4^4 \rightarrow H_6$ and 4-tr-Embedding $SOS_4^3 \rightarrow \frac{1}{2}H_{14}$

For even $n >$ and increasing sequence $\vec{a} = (a_1, a_2, \dots, a_k)$ of odd numbers from $[3, n-1]$, we introduce the *Generalised Chordal Ring* $GCR(n, \vec{a})$ as the graph obtained by adding to the cycle $C_{1, \dots, n}$, where each i is adjacent to $i-1$ and $i+1$ modulo n , the following edges:

- (1) if i is even, then i is adjacent to $i + a_l \pmod n$ for $1 \leq l \leq k$;
- (2) if i is odd, then i is adjacent to $i - a_l \pmod n$ for $1 \leq l \leq k$.

The cases $k = 1$ and 2 correspond to known topologies: the *Chordal Rings* and *Double Chordal Rings*, respectively. The Chordal Ring $GCR(n, a)$ is embeddable for $a = 1$ and 3 (being C_n and $Prism_{\frac{n}{2}}$, respectively), but for $a = 5$ and 7 , $GCR(n, a)$ (of diameter $d = 3$ and 4 , respectively) is not $(d-1)$ -tr.embeddable even for the smallest case $n = 2a$.

The results of our computations are summarized in the Conjecture below and Table 4, listing known embeddings, which are not covered by this Conjecture (ii).

See Figure 6 for $GCR(24, \{9, 11\})$, the smallest case in Conjecture (ii).

Conjecture 3. (checked for $v \leq 70, k \leq 5$ and $v \leq 200, k = 2, a_2 = a_1 + 2$)

(i) If $GCR(n, \vec{a})$ of diameter d is embeddable, then $n \equiv 0 \pmod 4$, $\vec{a} = \{a, a+2\}$ and embedding is into H_d .

(ii) For each $n \equiv 8 \pmod{16}, n \geq 24$, the Double Chordal Rings $GCR(n, (\frac{n}{2} - 3, \frac{n}{2} - 1))$ and $GCR(n, (\frac{n}{2} + 1, \frac{n}{2} + 3))$ have $d = \frac{n}{8} + 2$ and embed into H_d .

| | |
|------------------|------------------|
| 1 : (0,0,0,0,0) | 2 : (1,0,0,0,0) |
| 3 : (1,0,1,0,0) | 4 : (1,0,1,0,1) |
| 5 : (1,0,1,1,1) | 6 : (1,1,1,1,1) |
| 7 : (1,1,0,1,1) | 8 : (0,1,0,1,1) |
| 9 : (0,1,0,0,1) | 10 : (0,1,0,0,0) |
| 11 : (0,1,1,0,0) | 12 : (0,0,1,0,0) |
| 13 : (0,0,1,1,0) | 14 : (1,0,1,1,0) |
| 15 : (1,0,0,1,0) | 16 : (1,0,0,1,1) |
| 17 : (1,0,0,0,1) | 18 : (1,1,0,0,1) |
| 19 : (1,1,1,0,1) | 20 : (0,1,1,0,1) |
| 21 : (0,1,1,1,1) | 22 : (0,1,1,1,0) |
| 23 : (0,1,0,1,0) | 24 : (0,0,0,1,0) |

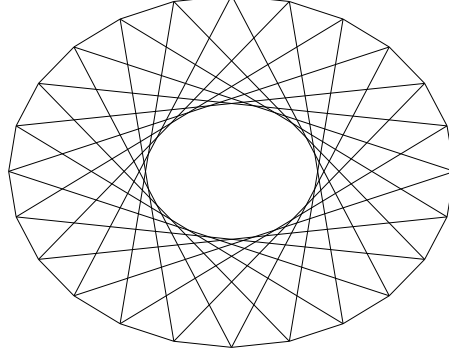


FIGURE 6. Embedding of Double Chordal Ring:
 $GCR(24, (9, 11)) \rightarrow H_5$

| n | \vec{a} | emb. into | n | \vec{a} | emb. into |
|-----|-----------|-----------|-----|-----------|-----------|
| 48 | (13,15) | H_7 | 60 | (21,23) | H_8 |
| 80 | (17,19) | H_9 | 84 | (25,27) | H_{10} |
| 96 | (33,35) | H_{11} | 112 | (29,31) | H_{11} |
| 120 | (21,23) | H_{11} | 120 | (37,39) | H_{13} |
| 132 | (45,47) | H_{14} | 140 | (57,59) | H_{12} |
| 144 | (33,35) | H_{13} | 156 | (49,51) | H_{16} |
| 160 | (61,63) | H_{13} | 168 | (25,27) | H_{13} |
| 168 | (57,59) | H_{17} | 176 | (45,47) | H_{15} |
| 180 | (37,39) | H_{14} | 192 | (61,63) | H_{19} |

TABLE 4. All known embeddings of $GCR(n, \vec{a})$, not covered by Conjecture 3 (ii). For two 120- and two 168-vertex graphs, \vec{a} is $(\frac{n}{4} - 9, \frac{n}{4} - 7)$, $(\frac{n}{4} + 7, \frac{n}{4} + 9)$ and $(\frac{n}{4} - 17, \frac{n}{4} - 15)$, $(\frac{n}{4} + 15, \frac{n}{4} + 17)$

9. REGULAR MAPS

A *map* is a 2-cell decomposition of a closed compact two-dimensional manifold, i.e., a decomposition of a 2-manifold into topological disks. A *regular map* is a map such that every *flag* (an incident vertex-edge-face triple) can be transformed into any other flag by a symmetry of the decomposition. The map of *type* $\{a, b\}$ is the regular map with degree a of vertices, having only b -gonal faces.

Each of five regular spherical maps, i.e., skeletons of Platonic polyhedra, are embeddable; cf., say, [6]. It holds

$$K_4 = \frac{1}{2}H_3 \simeq J(4, 1), \quad K_2^3 = H_3, \quad K_{2,2,2} = J(4, 2), \quad K_{2,2,2,2} = \frac{1}{2}H_4$$

for Tetrahedron, Cube, Octahedron, Hyperoctahedron, respectively. Icosahedron and Dodecahedron embed into $\frac{1}{2}H_6$, $\frac{1}{2}H_{10}$. See Figs. 7 and 5.

The *cubic Klein graph* is a 3-regular graph of diameter 6 with 56 vertices, which is the skeleton of the *Klein map*, a symmetric tessellation of a genus 3 surface by 24 heptagons. Neither it, nor its dual are embeddable. The cubic Klein and Dick graphs are Cayley graphs. See their dual (on genus 3 surface) on Fig. 1.

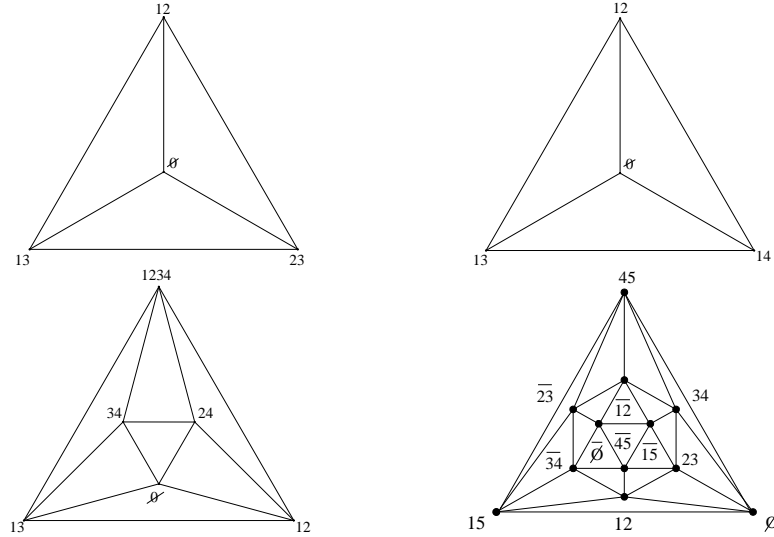


FIGURE 7. Embeddings of Tetrahedron (two), Octahedron and Icosahedron into 3-, 4-, 4- and 6-half-cube, respectively

The *Dyck graph* is a 3-regular graph of diameter 5 with 32 vertices, which is the skeleton of the *Dyck map*, a symmetric tessellation of a genus-3 surface by 12 octagons. Neither it, nor $K_{4,4,4}$ (its dual for this tiling) are embeddable. The Dyck graph is toroidal; the skeleton of its dual on the torus \mathbb{T}^2 is the *Shrikhande graph*, which embeds into $\frac{1}{2}H_6$. The Shrikhande graph can be constructed as a Cayley graph on $\mathbb{Z}_4 \times \mathbb{Z}_4$ with two vertices being adjacent if the difference is in $\{\pm(1, 0), \pm(0, 1), \pm(1, 1)\}$.

The Dyck graph admits a 4-tr.embedding into H_6 ; see it on Table 5. Each column of this 32×6 binary matrix $X = ((x_{ij}))$ has exactly 16 ones. Let $X' = ((1 - x_{ij}))$. Clearly, the 32-sets of rows of X and X' form together the 64 vertices of H_6 . For every vertex v of the Dyck graph, its stabilizer has two orbits, of sizes 3 and 1, of antipodal (i.e., at distance 5) points, say, $\{v'\}$. The distance matrices D of the Dyck graph and $D(X)$ (Hamming pairwise distances of rows of X) differ only in 16 entries: 16 distances of the form $d(v, v')$ are 5 in D , but became 3 in $D(X)$.

We analyzed all regular maps from [4] up to genus 13 and found embeddings of skeletons for many of them. We do not take just the maps occurring there, but also the maps obtained from them by the so-called *Wythoff construction* (see [5] for an exposition). In our context, the Wythoff construction takes a map M , a non-trivial subset S of $\{0, 1, 2\}$ and returns another map $W_S(M)$. We embedding

Conjecture 4. (checked for all maps of genus $g \leq 13$)

- (i) For any $g \geq 3$, there exist a unique map M of genus g and type $\{4, 4g\}$ such that its skeleton is the cycle $C_{2g} \rightarrow H_g$.
- (ii) For any $g \geq 2$ there exist a unique map M of genus g and type $\{4g, 4g\}$ such that its skeleton is a cycle $C_{2g} \rightarrow H_g$.
- (iii) For any $g \geq 2$, there exist a unique map M of genus g and type $\{4, 2g + 2\}$ such that its skeleton is a cycle $C_{2g+2} \rightarrow H_{g+1}$, the dual skeleton is C_4 and the map

| | | | |
|---------------|---------------|---------------|---------------|
| (0,0,0,0,0,0) | (1,0,0,0,0,0) | (1,0,0,1,0,0) | (1,0,0,1,0,1) |
| (1,1,0,1,0,1) | (0,1,0,1,0,1) | (0,1,0,0,0,1) | (0,1,0,0,0,0) |
| (1,0,0,0,1,0) | (1,1,0,0,1,0) | (1,1,1,0,1,0) | (1,1,1,1,1,0) |
| (1,1,1,1,0,0) | (1,0,1,1,0,0) | (0,0,1,0,0,0) | (0,0,1,0,0,1) |
| (0,0,1,0,1,1) | (1,0,1,0,1,1) | (1,0,0,0,1,1) | (0,1,0,0,1,0) |
| (0,1,0,1,1,0) | (0,1,1,1,1,0) | (0,0,1,1,1,0) | (0,0,1,1,0,0) |
| (0,1,1,0,0,1) | (1,1,1,0,0,1) | (1,1,1,0,1,1) | (0,1,0,1,1,1) |
| (0,0,0,1,1,1) | (0,0,1,1,1,1) | (1,1,1,1,0,1) | (1,0,0,1,1,1) |

TABLE 5. 4-tr.embedding of the Dyck graph into H_6

$W_{\{0,2\}}(M)$ has $(8g+8)$ -vertex skeleton of diameter $g+3$, that is embeddable into H_{g+3} .

(iv) For any $g \geq 2$, there exist a unique map M of genus g and type $\{2g+1, 4g+2\}$ such that the skeleton of $W_{\{1\}}(M)$ is a cycle $C_{2g+1} \rightarrow \frac{1}{2}H_{2g+1}$ and the $W_{\{0,1\}}(M)$ has $(4g+2)$ -vertex skeleton of diameter $g+1$, that is embeddable into $\frac{1}{2}H_{2g+3}$.

(v) For any $g \geq 2$, there exist a unique map M of genus g and type $\{2g+2, 2g+2\}$ such that $W_{\{0,1\}}(M)$ has $(4g+4)$ -vertex skeleton of diameter $g+2$, that is embeddable into H_{g+2} .

REFERENCES

- [1] A. Andoni, M. Deza, A. Gupta, P. Indyk and S. Raskhodnikova, *Lower Bounds for Embedding Edit Distance into Normed Spaces*, Proceedings of SODA'03 (ACM-SIAM Symposium on Discrete Algorithms, January 2003, Baltimore).
- [2] D. Avis, *Hypermetric spaces and the Hamming cone*, Canadian Journal of Mathematics **33** (1981) 795–802.
- [3] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer, 1989.
- [4] M. Conder and P. Dobsányi, *Determination of all Regular Maps of Small Genus*, Journal of Combinatorial Theory, Series B **81** (2001) 224–242.
- [5] M. Deza, M. Dutour Sikirić and S. Shpectorov, *Hypercube Embeddings of Wythoffians*, Ars Mathematica Contemporanea **1** (2008) 99–111.
- [6] M. Deza, V.P. Grishukhin and M.I. Shtogrin, *Scale-isometric polytopal graphs in hypercubes and cubic lattices*, Imperial College Press, 2004.
- [7] M. Deza and M. Laurent, *Geometry of cuts and metrics*, Algorithms and Combinatorics 15, Springer, 1997.
- [8] M. Deza and S. Shpectorov, *Recognition of the l_1 -graphs with complexity $O(nm)$ and football in hypercube*, in Special Issue *Discrete Metric Spaces*, European Journal of Combinatorics **17** (1996) 279–289.
- [9] M.-C. Heydemann, *Cayley graphs and interconnection networks*, in "Graph Symmetry: Algebraic Methods and Applications", ed. by G. Hahn and G. Sabidussi, Springer, 1997.
- [10] S. Klavzar, *Structure of Fibonacci cubes: a survey*, IMFM Preprint Series (Ljubljana, Slovenia: Institute of Mathematics, Physics and Mechanics) **49** (2011) 1150.
- [11] J. Koolen, *On metric properties of regular graphs*, Master's thesis, Technische Universiteit Eindhoven, 1990.
- [12] G.Kotsis, *Interconnection Topologies and Routing for Parallel Processing Systems*, ACPC Technical Reports Series, ACPC/TR92-19, 1992.
- [13] *Open problems on embeddings of finite metric spaces* edited by J. Matoušek, 2011, <http://kam.mff.cuni.cz/~matousek/metrop.ps>
- [14] F.P. Preparata, J. Vuillemin, *The cube-connected cycles: a versatile network for parallel computation*, Communications of the ACM **24** (5): 300309, 1981.
- [15] N.D. Roussopoulos, *A $\max\{m, n\}$ algorithm for determining the graph H from its line graph G* , Information Processing Letters (2): 108–112, 1973.

- [16] J. Subercaze, C. Gravier and F. Laforest, *On metric embedding for boosting semantic similarity computations*, Association of Computational Linguistics, Jul 2015, Beijing, China. hal-01166163.
- [17] M.E. Tylkin (=M. Deza), *On Hamming geometry of unitary cubes*, Soviet Physics. Doklady **5** (1960) 40–943.
- [18] D.B. West, *Introduction to Graph Theory*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 2000.

MATH. DEPT., KING ABDULAZIZ UNIVERSITY, JEDDAH 21589, SAUDI ARABIA
E-mail address: `adelnife2@yahoo.com`

MATH. DEPT., KING ABDULAZIZ UNIVERSITY, JEDDAH 21589, SAUDI ARABIA
E-mail address: `alhazmih@yahoo.com`

MATH. DEPT, FACULTY OF SCIENCE, RABIGH CAMPUS, KING ABDULAZIZ UNIVERSITY, RABIGH,
SAUDI ARABIA
E-mail address: `shakir50@rediffmail.com`

MICHEL DEZA, ÉCOLE NORMALE SUPÉRIEURE, 75005 PARIS, FRANCE
E-mail address: `Michel.Deza@ens.fr`

MATHIEU DUTOUR SIKIRIĆ, RUDJER BOSKOVIĆ INSTITUTE, BIJENICKA 54, 10000 ZAGREB,
CROATIA, FAX: +385-1-468-0245
E-mail address: `mathieu.dutour@gmail.com`

PATRICK SOLÉ, TÉLÉCOM PARIS TECH, 46 RUE BARRAULT, 75013 PARIS, FRANCE
E-mail address: `patrick.sole@telecom-paristech.fr`